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SLOWLY VARYING HIGH-FREQUENCY STRESS-STRAIN STATES IN IMMERSED SHELLS*

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The asymptotic integration method is used to derive two-dimensional equations that describe, near the cutoff frequency, the slowly varying component of the stress-strain state of a thin elastic shell immersed in an infinite compressible liquid. The effect of the fluid on different types of high-frequency, long-wave vibrations of the shell is established. Applications of the equations to hydroacoustic problems are discussed for the case of a circular cylindrical shell.

The asymptotic integration method has been previously used to derive two-dimensional equations describing slowly varying high-frequency stress-strain states in an elastic layer superimposed on an acoustic half-space /1, 2/ and in a "dry" shell (without contact with a liquid) /3/. Slowly varying high-frequency stress-strain states in dry plates and shells are studied by the variational approach in /4, 5/.

1. *Basic relationships*. Consider a closed convex thin elastic shell immersed in an infinite compressible liquid which performs harmonic oscillations of the form $e^{-i\omega t}$. The radius-vector of a point in three-dimensional space is represented as the sum

$$\mathbf{P}(\alpha_1, \alpha_2, \alpha_3) = \mathbf{M}(\alpha_1, \alpha_2) + \alpha_3\mathbf{n} \tag{1.1}$$

where $\mathbf{M}(\alpha_1, \alpha_2)$ is the radius-vector of the points of the shell middle surface, $\mathbf{n}$ is the unit normal vector to the shell and α_3 is the distance along the normal from the midsurface. We assume that the midsurface is related to the lines of curvature and the vector equality (1.1) accordingly defines a triorthogonal coordinate system.

We will write out the basic relationships of the problem using the following notation: σ_{mn} and v_k $(m, n, k = 1, 2, 3)$ are the stresses and the displacements of the elastic medium forming the shell, φ is the liquid displacement potential, $Q_k^\pm$ $(k = 1, 2, 3)$ are the loads applied to the shell faces, $2h$ is the shell thickness: R_1, R_2 and R, respectively, are the principal radii of curvature and the characteristic radius of curvature of the shell midsurface, E is Young's modulus, ν is Possion's ratio, c_1 and c_2 are the velocities of propagation of shear waves and compressional-dilatational waves in the shell material, c_0 is the velocity of sound in the liquid and ρ_0, ρ, respectively, are the density of the liquid and of the shell material.

The dynamic equations of elasticity theory in the region $-h \leqslant \alpha_3 \leqslant h$ filled by the shell are written in the form

$$\begin{gathered}
H_i^{-1}\sigma_{ii,i} + H_j^{-1}\sigma_{ij,j} + \sigma_{i3,3} + (H_iH_j)^{-1}H_{j,i}(\sigma_{ii} - \sigma_{jj}) + \\
2(H_iH_j)^{-1}H_{i,j}\sigma_{ij} + (2H_i^{-1}H_{i,3} + H_j^{-1}H_{j,3})\sigma_{i3} + \rho\omega^2 v_i = 0 \\
H_i^{-1}\sigma_{3i,i} + H_j^{-1}\sigma_{3j,j} + \sigma_{33,3} - H_i^{-1}H_{i,3}\sigma_{ii} - H_j^{-1}H_{j,3}\sigma_{jj} + \\
(H_iH_j)^{-1}[(H_iH_j)_{,3}\sigma_{33} + H_{j,i}\sigma_{3i} + H_{i,j}\sigma_{3j}] + \rho\omega^2 v_3 = 0 \\
\sigma_{ii} = 2E_* [\beta_1 (v_{3,3} + H_j^{-1}v_{j,j} + H_i^{-1}H_j^{-1}H_{j,i}v_i + H_j^{-1}H_{j,3}v_3) + \\
(1 + \beta_1)H_i^{-1}(v_{i,i} + H_{i,3}v_3 + H_j^{-1}H_{i,j}v_j)] \\
\sigma_{33} = 2E_* [\beta_1 (H_i^{-1}v_{i,i} + H_j^{-1}v_{j,j} + H_i^{-1}H_j^{-1}H_{j,i}v_i + H_i^{-1}H_j^{-1}H_{i,j}v_j + \\
H_i^{-1}H_{i,3}v_3 + H_j^{-1}H_{j,3}v_3) + (1 + \beta_1)v_{3,3}] \\
\sigma_{i3} = E_* (H_i^{-1}v_{3,i} + v_{i,3} - H_i^{-1}H_{i,3}v_i) \\
\sigma_{ij} = E_* [H_j^{-1}v_{i,j} + H_i^{-1}v_{j,i} - (H_iH_j)^{-1}(H_{i,j}v_i + H_{j,i}v_j)] \\
H_i = A_i(1 + \alpha_3/R_i),\ i, j = 1, 2;\ i \neq j;\ \beta_1 = \nu/(1 - 2\nu) \\
E_* = {}^1/_2 E/(1 + \nu),\ f_{,k} = \partial f/\partial\alpha_k,\ k = 1, 2, 3
\end{gathered} \tag{1.2}$$

Here A_i are the coefficients of the first quadratic form of the midsurface.

The oscillations of the liquid in the region exterior to the shell $\alpha_3 \geqslant h$ are described by the Helmholtz equation

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$$\Delta_3\varphi + (\omega^2/c_0^2)\varphi = 0 \tag{1.3}$$

where Δ_3 is the three-dimensional Laplace operator. The boundary conditions on the shell faces have the form

$$\sigma_{3i}(\alpha_1, \alpha_2, \pm h) = Q_i^{\pm} \ (i = 1, 2), \quad \sigma_{33}(\alpha_1, \alpha_2, -h) = Q_3^- \tag{1.4}$$
$$\sigma_{33}(\alpha_1, \alpha_2, h) = Q_3^+ - \rho_0\omega^2\varphi(\alpha_1, \alpha_2, h), \ v_3(\alpha_1, \alpha_2, h) = \partial\varphi/\partial\alpha_3\,|_{\alpha_3=h}$$

The potential φ is required to satisfy the radiation condition

$$\partial\varphi/\partial\,|\mathbf{P}| - i(\omega/c_0)\varphi = o(|\mathbf{P}|^{-1}) \quad \text{as} \quad |\mathbf{P}| \to \infty \tag{1.5}$$

We will assume that the relative shell half-thickness $\eta = h/R$ and the relative liquid impedance $\varepsilon = c_0\rho_0/(c_2\rho)$ are small parameters. The relationship between these parameters is taken in the form

$$\varepsilon = \eta^t\varepsilon_0 \ (t > 0, \ \varepsilon_0 \sim \eta^0) \tag{1.6}$$

We will consider the frequency range in which

$$\omega R/c_2 = \eta^{-1}\mu \ (\mu \sim \eta^0) \tag{1.7}$$

Let us stretch the scale of the independent variables in (1.2) by the formulas

$$\alpha_i = R\eta^q\xi_i, \quad \alpha_3 = R\eta\zeta \tag{1.8}$$

Here q is the exponent of variability of the required stress-strain state by the variables α_i $(i = 1, 2)$; we assume that differentiation with respect to the variables ξ_i, ζ does not change the order of the original variables.

Consider the stress-strain (SSS) of a shell with exponent of variability $q < 1$, which in the range (1.7) will be called slowly varying high-frequency stress-strain states (SV HF SSS). We will show that SV HF SSS occur in narrow neighbourhoods of the cutoff frequencies and that two types of such SSS exist. Type 1 SSS correspond to quasitransverse oscillations of the shell when $v_3 \gg v_i$ $(i = 1, 2)$ and type 2 SSS correspond to quasitangential oscillations for which conversely $v_i \gg v_3$. We accordingly expect that the effect of the liquid on the oscillations of the shell depends on the type of SSS considered.

Before proceeding to study SV HF SSS, we will apply assumptions (1.6)-(1.8) to express the potential φ in the force boundary condition on the contact surface in terms of the parameters of the shell SSS.

2. Asymptotic representation of the pressure of the liquid on the shell. Let us first consider an auxiliary Dirichlet problem for the Helmholtz Eq.(1.3) in the region $\alpha_3 \geqslant h$ with the boundary condition

$$\varphi(\alpha_1, \alpha_2, h) = \psi(\alpha_1, \alpha_2) \tag{2.1}$$

and radiation condition (1.5) at infinity. In (2.1), $\psi(\alpha_1, \alpha_2)$ is a given function such that $R\partial\psi/\partial\alpha_i \sim \eta^{-q}\psi$ $(i = 1, 2;\ q < 1)$.

Let us derive an asymptotic representation of the solution of this problem in a thin layer of the liquid of width $\gamma = \alpha_3/h - 1 \sim \eta^0$ near the wall. The displacement potential is specified in the form

$$\varphi = \exp(ic\mu\gamma)[\varphi_0(\alpha_1, \alpha_2) + \gamma\eta\varphi_1(\alpha_1, \alpha_2) + O(\eta^{2-2q})], \ c = c_2/c_0 \tag{2.2}$$

Substituting (2.2) into (1.3) and (2.1), we obtain

$$\varphi_0 = \psi, \ \varphi_1 = -{}^1/_2R\psi\,(1/R_1 + 1/R_2) \tag{2.3}$$

This result enables us to express the pressure of the liquid on the shell in the third boundary condition in (1.4) in terms of the normal displacement of the shell face. Indeed, substituting the representation (2.2) into the imperviousness condition (the last condition in (1.4)) and using relationships (2.3), we obtain

$$\varphi(\alpha_1, \alpha_2, h) = -\frac{ihv_3(\alpha_1, \alpha_2, h)}{c\mu}\left[1 - \frac{i\eta}{2c\mu}\left(\frac{R}{R_1} + \frac{R}{R_2}\right) + O(\eta^{2-2q})\right] \tag{2.4}$$

The term with the multiplier η in the bracketed expression represents the effect of shell curvature. We shall see in what follows that this effect may be quite substantial in some cases for $q < {}^1/_2$. The presence of this term distinguishes the representation (2.4) from the common hydroacoustic approximation obtained in the framework of the piston theory*.

*Gol'denveizer A.L. and Radovinskii A.L., Asymptotic Analysis of Oscillations and Radiation of a Shell in a Liquid, Preprint 275, Moscow, Akad. Nauk SSSR, Inst. Probl. Mekhan., 1986.

3. Two-dimensional equations of type 1 SV HF SSS. SV HF SSS observed near the cutoff frequencies $\Lambda = \pi m/(2\beta)$ $(\beta = c_2/c_1 = [(1 - 2\nu)/(2 - 2\nu)]^{1/2},\ m \in \mathbb{N},\ m \sim \eta^0)$ will be called type 1 SV HF SSS. The frequencies Λ are the eigenfrequencies of the compressional-dilatational modes of a transverse fibre in a dry shell. The neighbourhoods of these frequencies are defined by the asymptotic relationship.

$$\mu - \Lambda \sim \eta^b \ (b > 0) \tag{3.1}$$

where b is the deviation exponent.

Let us examine the properties of these SSS for free oscillations $(Q_k^{\pm} = 0\ (k = 1, 2, 3))$. We define their asymptotic behaviour in the form

$$v_3 = hu_3,\ \sigma_{kk} = E_* s_{kk};\ v_i = h\eta^{1-q}u_i,\ \sigma_{i3} = E^*\eta^{1-q}s_{i3},\ \sigma_{ij} = E_*\eta^{2-2q}s_{ij} \tag{3.2}$$

Here and below we assume that the dimensionless quantities $u_k,\ s_{kk},\ s_{i3},\ s_{ij}$ $(i, j = 1, 2;\ i \neq j;\ k = 1, 2, 3)$ are of the same asymptotic order. The existence of SSS with asymptotic behaviour (3.2) in the neighbourhoods (3.1) of the cutoff frequencies Λ will be proved as follows.

Let us derive approximate two-dimensional equations (independent of the transverse coordinate ζ) which describe type 1 SV HF SSS in an immersed shell. We will first simplify Eqs.(1.2) and the boundary conditions (1.4) (using (2.4)) by dividing the required SSS into symmetric and antisymmetric components relative to the shell midsurface. The shell displacements can accordingly be represented in the form of a sum

$$u_k = u_k^0 + \eta^s u_k^1,\ s = \min(1, t) \tag{3.3}$$

The variables with the superscript 0 relate to the asymptotically principal symmetric (antisymmetric) SSS component, and the variables with the superscript 1 relate to the asymptotically secondary antisymmetric (symmetric) component. We assume that u_k^0, u_k^1 are of the same asymptotic order. The representation (3.3) is made possible by the structure of the relationships (1.2), (1.4), (1.6), (2.4) and will be justified below.

Let us now substitute formulas (1.6)-(1.8), (2.4), (3.2), (3.3) into (1.2), (1.4). We will express the stresses in terms of displacements and omit the terms which are ignorable in the first approximation for constructing the final two-dimensional equations. Using the fact that u_k^0, u_k^1 are even (odd) with respect to ζ, we obtain, after some reduction, the system of equations

$$\begin{gathered}\partial^2\mathbf{u}_\tau^0/\partial\zeta^2 + \mu^2\mathbf{u}_\tau^0 + (1 - 2\nu)^{-1}\operatorname{grad}\partial u_3^0/\partial\zeta + O(\eta^r) = 0 \\ \partial^2 u_3^0/\partial\zeta^2 + \mu^2\beta^2 u_3^0 + \eta^{2-2q}\beta^2[(1 - 2\nu)^{-1}\operatorname{div}\mathbf{u}_\tau^0 + \Delta u_3^0] - \\ \eta^2\varkappa_2(u_3^0 + \zeta\partial u_3^0/\partial\zeta) + \eta^{1+s}\varkappa_1\partial u_3^1/\partial\zeta + O(\eta^{3-2q+s}) = 0 \\ \partial^2 u_3^1/\partial\zeta^2 + \mu^2\beta^2 u_3^1 + \eta^{1-s}\varkappa_1\partial u_3^0/\partial\zeta + O(\eta^{2-2q}) = 0 \\ r = \min(2 - 2q, 1 + t),\ \mathbf{u}_\tau = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 \\ \varkappa_j = R^j(1/R_1^j + 1/R_2^j),\quad j = 1, 2\end{gathered} \tag{3.4}$$

with the boundary conditions for $\zeta = 1$

$$\begin{gathered}\partial\mathbf{u}_\tau^0/\partial\zeta + \operatorname{grad} u_3^0 + O(\eta^{1+s}) = 0 \\ (1 + \beta_1)\partial u_3^0/\partial\zeta + \beta_1(\eta^{2-2q}\operatorname{div}\mathbf{u}_\tau^0 - \eta^2\varkappa_2 u_3^0 + \eta^{1+s}\varkappa_1 u_3^1) - \\ \eta^{1+t}[\varepsilon_0\varkappa_1/(8c)]u_3^0 - {}^1/_4 i\eta^t\varepsilon_0\mu(u_3^0 + \eta^s u_3^1) + O(\eta^{r+t} + \eta^{3-2q+s}) = 0 \\ (1 + \beta_1)\partial u_3^1/\partial\zeta - {}^1/_4 i\eta^{t-s}\varepsilon_0\mu(u_3^0 + \eta^s u_3^1) + \eta^{1-s}\varkappa_1 u_3^0[\beta_1 - \eta^t\varepsilon_0/(8c)] + \\ O(\eta^r) = 0\end{gathered} \tag{3.5}$$

Here $\mathbf{i}_1$ and $\mathbf{i}_2$ are the unit vectors of the orthogonal system of coordinates chosen on the midsurface. All the differential operators occurring in (3.4), (3.5) act on the shell midsurface.

Let us consider the case when the SSS defined by u_k^0 is antisymmetric with respect to ζ. Then in (3.1) $m = 2n,\ n \in \mathbb{N}$. From the second equation in (3.4) we obtain the representation

$$u_3^0 = w(\xi_1, \xi_2)\cos\beta\mu\zeta + \eta^r u_{3r}^0(\xi_1, \xi_2, \zeta) \tag{3.6}$$

where $w(\xi_1, \xi_2)$ and $u_{3r}^0(\xi_1, \xi_2, \zeta)$ are unknown functions. Substituting expression (3.6) into (3.4) and in the first and third boundary conditions in (3.5), we obtain with error $O(\eta^r + \eta^b)$

$$\begin{gathered}\mathbf{u}_\tau^0 = \frac{\operatorname{grad} w}{\Lambda}\left(\frac{\sin\beta\Lambda\zeta}{\beta} - \frac{2\cos\beta\Lambda\sin\Lambda\zeta}{\cos\Lambda}\right) \\ u_3^1 = 2\beta w\left\{\eta^{1-s}\varkappa_1\left[\frac{1}{2\Lambda}(1 - \beta_1)\sin\beta\Lambda\zeta - \frac{1}{4\beta}\zeta\cos\beta\Lambda\zeta + \right.\right. \\ \left.\left.\frac{1}{8}\eta^t\varepsilon_0\left(\frac{1}{\Lambda c} - i\right)\sin\beta\Lambda\zeta\right] + \frac{1}{4}i\eta^{t-s}\varepsilon_0\sin\beta\Lambda\zeta\right\} \\ u_{3r}^0 = \eta^{2-2q-r}A_{2-2q-r} + \eta^{1+t-r}A_{1+t-r}\end{gathered} \tag{3.7}$$

$$A_{2-2q-r} = -\frac{\Delta w}{\Lambda}\left(\frac{\zeta \sin \beta\Lambda\zeta}{2\beta} + \frac{2\cos\beta\Lambda\cos\Lambda\zeta}{\Lambda\cos\Lambda}\right) - \eta^{2q} w \times$$
$$\left\{\left[\varkappa_1^2\left(\frac{1}{4} - \frac{\beta_1}{1+\beta_1}\right) - \frac{1}{2}\varkappa_2\right]\frac{\zeta\sin\beta\Lambda\zeta}{\beta\Lambda} - \frac{1}{4}\zeta^2\left(\frac{1}{2}\varkappa_1^2 + \varkappa_2\right)\cos\beta\Lambda\zeta\right\}$$
$$A_{1+t-r} = \frac{1}{4}\varkappa_1\varepsilon_0\beta w\zeta\sin\beta\Lambda\zeta\left[i + \frac{1}{2}\varkappa_1\eta\left(\frac{1}{\Lambda c} - i\right)\right]$$

Using formulas (3.7), we can show that all the quantities in relationships (3.2), (3.3) are of the same asymptotic order. For $t > 1$ they reduce with an additional error $O(\eta^{t-1})$ to the corresponding formulas for a dry shell /3/.

We substitute (3.7) into the second boundary condition (3.5). Collecting similar terms and dropping asymptotically secondary quantities, we obtain

$$\eta^{2-2q} T^* \Delta w + [\eta^2 T_R^* + {}^1/_2 i\eta^t \varepsilon_0 + {}^1/_4 \eta^{1+t}\varepsilon_0\varkappa_1((\Lambda c)^{-1} - i) + (\mu - \Lambda)]w + O[\eta^{2-4q} + (\eta^r + \eta^b)(\eta^t + \eta^b)] = 0 \tag{3.8}$$
$$T^* = \frac{1}{2\Lambda\beta^2} - \frac{4\operatorname{tg}\Lambda}{\Lambda^2}, \quad T_R^* = -\frac{1}{2\Lambda}\left[\varkappa_2(3-\beta_1) + \frac{1}{2}\varkappa_1^2(1+\beta_1)\right]$$

The transformations applied in deriving the two-dimensional Eq.(3.8) hold when $T^* \sim \eta^0$. If this condition is not satisfied, additional analysis is required /2/. Note that in addition to the term $O(\eta^t)$ we retain also the term $O(\eta^{1+t})$ in (3.8). The imaginary part of the term $O(\eta^{1+t})$ obviously can be omitted to a first approximation compared with the purely imaginary term $O(\eta^t)$. The real part of the term $O(\eta^{1+t})$, however, may have a significant effect in some problems. An appropriate example is given in Sect.5.

Let us change in Eq.(3.8) to the original dimensional coordinates α_i on the shell midsurface, using the notation $v_3^a(\alpha_1, \alpha_2) = hw(\alpha_1, \alpha_2)$. Omitting the O-term, we write

$$T\Delta v_3^a + \left[T_R + \frac{1}{2}i\varepsilon + \frac{1}{4}\varepsilon h\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{1}{\Lambda c} - i\right) + \left(\frac{\omega h}{c_2} - \Lambda\right)\right]v_3^a = 0 \tag{3.9}$$
$$T = h^2 T^*, \quad T_R = -\frac{2h^2}{\Lambda}\left[\left(1 - \frac{1}{16\beta^2}\right)\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right) + \frac{1}{8\beta^2 R_1 R_2}\right]$$

If $\varepsilon = 0$, then (3.9) reduces to the equation describing the SV HF SSS of a dry shell /3/. The case when the displacements u_k^0 define a symmetric SSS (relative to the shell midsurface) is analysed similarly. Eq.(3.9) remains applicable, apart from the replacement of tg with $-\operatorname{ctg}$ in the expression for the coefficient T and n with $n - {}^1/_2$ in the expression for Λ.

In the case of forced oscillations, the right-hand side of Eq.(3.9) acquires an additional term which represents the action of the forces on the shell faces. It is identical with the corresponding term for the dry shell and has the form

$$P_3 = \pm\frac{2(-1)^{n+1}h}{\Lambda}\left[F_3 + \frac{h\Phi_3}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{h\operatorname{div}\mathbf{F}_\tau}{\Lambda}\left\{\begin{matrix}\operatorname{tg}\Lambda \\ -\operatorname{ctg}\Lambda\end{matrix}\right\}\right] \tag{3.10}$$
$$F_3 = {}^1/_4(Q_3^+ \mp Q_3^-)/E_*, \quad \Phi_3 = {}^1/_4(Q_3^+ \pm Q_3^-)/E_*$$
$$F_j = {}^1/_2(Q_j^+ \pm Q_j^-)/E_*, \ j = 1, 2, \ \mathbf{F}_\tau = F_1\mathbf{i}_1 + F_2\mathbf{i}_2$$

The upper (lower) signs and upper (lower) expressions in braces in (3.10) correspond to the case when the asymptotically principal component of the shell SSS is antisymmetric (symmetric) relative to the shell midsurface.

4. Two-dimensional equations of type 2 SV HF SSS. Type 2 SV HF SSS are observed in the neighbourhoods (3.1) of the cutoff frequencies $\Lambda = \pi m/2$ $(m \in \mathrm{N}, m \sim \eta^0)$, which are the eigenfrequencies of the shear modes of a transverse fibre in the shell. The asymptotic behaviour of these SSS is given by

$$v_i = hu_i, \ v_3 = h\eta^{1-q}u_3 \tag{4.1}$$
$$\sigma_{i3} = E_* s_{i3}; \ \sigma_{kk} = E_*\eta^{1-q}s_{kk}, \ \sigma_{ij} = E_*\eta^{1-q}s_{ij}$$

The partition of the shell displacements into symmetric and antisymmetric components relative to the midsurface is defined in this case by the relationships

$$u_i = u_i^0 + \eta^{s_1}u_i^1, \ u_3 = u_3^0 + \eta^s u_3^1, \ s_1 = \min(1, 2 - 2q + t) \tag{4.2}$$

Substituting (1.6)-(1.8), (2.4), (4.1), and (4.2) into (1.2), (1.4) and using the same algebra as in Sect.3, we obtain the system of equations

$$\partial^2 u_3^0/\partial\zeta^2 + \mu^2\beta^2 u_3^0 + (2-2\nu)^{-1}\operatorname{div}\partial\mathbf{u}_\tau^0/\partial\zeta + O(\eta^{2-2q}) = 0 \tag{4.3}$$
$$\partial^2\mathbf{u}_\tau^0/\partial\zeta^2 + \mu^2\mathbf{u}_\tau^0 + \eta^{2-2q}[(1-2\nu)^{-1}\operatorname{grad}(\partial u_3^0/\partial\zeta + \operatorname{div}\mathbf{u}_\tau^0) + \Delta\mathbf{u}_\tau^0] -$$
$$\eta^2(\varkappa_1 L_1\mathbf{u}_\tau^0 + \varkappa_2\zeta\partial\mathbf{u}_\tau^0/\partial\zeta) + \eta^{1+s_1}\varkappa_1\partial\mathbf{u}_\tau^1/\partial\zeta + O(\eta^{2-2q+s}) = 0$$
$$\partial^2\mathbf{u}_\tau^1/\partial\zeta^2 + \mu^2\mathbf{u}_\tau^1 + \eta^{1-s_1}\varkappa_1\partial\mathbf{u}_\tau^1/\partial\zeta + O(\eta^{2-2q+s-s_1}) = 0$$
$$L_j\mathbf{u}_\tau = R^j(R_1^{-j}u_1\mathbf{i}_1 + R_2^{-j}u_2\mathbf{i}_2),\ j = 1, 2$$

with the boundary conditions for $\zeta = 1$

$$(1+\beta_1)\partial u_3^0/\partial\zeta + \beta_1\operatorname{div}\mathbf{u}_\tau^0 - {}^1/_4 i\eta^t\varepsilon_0\mu u_3^0 + O(\eta^{2s}) = 0 \tag{4.4}$$
$$\partial\mathbf{u}_\tau^0/\partial\zeta + \eta^{2-2q}\operatorname{grad}u_3^0 + \eta^2 L_2\mathbf{u}_\tau^0 - \eta^{1+s_1}L_1\mathbf{u}_\tau^1 + O(\eta^{3-2q+s}) = 0$$
$$\partial\mathbf{u}_\tau^1/\partial\zeta - \eta^{1-s_1}L_1\mathbf{u}_\tau^0 + O(\eta^{2-2q+s-s_1}) = 0$$

Let us examine the case when the SSS defined by u_k^0 $(k = 1, 2, 3)$ is antisymmetric with respect to ζ. The expression for the cutoff frequencies takes the form $\Lambda = {}^1/_2\pi(n - {}^1/_2)$ $(n \in \mathrm{N},\ n \sim \eta^0)$. From the second equation in (4.3), we obtain

$$\mathbf{u}_\tau^0 = \boldsymbol{\psi}(\xi_1, \xi_2)\sin\mu\zeta + \eta^{2-2q}\mathbf{u}_{\tau r}^0(\xi_1, \xi_2, \zeta) \tag{4.5}$$

where $\boldsymbol{\psi}(\xi_1, \xi_2)$, $\mathbf{u}_{\tau r}^0(\xi_1, \xi_2, \zeta)$ are unknown vectors.

Substituting expression (4.5) into (4.3) and into the first and third boundary conditions in (4.4), we obtain

$$u_3^0 = \frac{\operatorname{div}\boldsymbol{\psi}}{\Lambda}\left[\cos\Lambda\zeta - \frac{\beta\cos\beta\Lambda\zeta}{\sin\beta\Lambda}(2\sin\Lambda - i\eta^t\varepsilon_0\beta\operatorname{ctg}\beta\Lambda)\right] + \tag{4.6}$$
$$O(\eta^{2-2q} + \eta^{2s} + \eta^b)$$
$$\mathbf{u}_\tau^1 = -\eta^{1-s_1}\boldsymbol{\psi}[(L_1 + {}^1/_2\varkappa_1)\Lambda^{-1}\cos\Lambda\zeta + {}^1/_2\varkappa_1\zeta\sin\Lambda\zeta] +$$
$$O(\eta^{2-2q+s-s_1} + \eta^b)$$
$$\mathbf{u}_{\tau r}^0 = \frac{\Delta\boldsymbol{\psi}\zeta\cos\Lambda\zeta}{2\Lambda} - \frac{\operatorname{grad}\operatorname{div}\boldsymbol{\psi}\sin\beta\Lambda\zeta}{\Lambda^2\sin\beta\Lambda}(2\sin\Lambda - i\eta^t\varepsilon_0\beta\operatorname{ctg}\beta\Lambda) +$$
$$\frac{1}{4}\boldsymbol{\psi}\zeta\left(\frac{1}{2}\varkappa_1^2 + \varkappa_2\right)\left(\frac{\cos\Lambda\zeta}{\Lambda} + \zeta\sin\Lambda\zeta\right) + O(\eta^{2-2q} + \eta^{2s} + \eta^b)$$

Substituting (4.6) into the second boundary condition of (4.4) and omitting asymptotically secondary terms, we obtain a two-dimensional system of equations for $\boldsymbol{\psi}$,

$$\eta^{2-2q}[{}^1/_2\Lambda^{-1}\Delta\boldsymbol{\psi} + (B^* + i\eta^t\varepsilon_0 B_0^*)\operatorname{grad}\operatorname{div}\boldsymbol{\psi}] + [\eta^2 B_R^* + (\mu - \tag{4.7}$$
$$\Lambda)]\boldsymbol{\psi} + O(\eta^{4-4q} + \eta^{2b} + \eta^{2-2q+2t}) = 0$$
$$B^* = \frac{4\beta\operatorname{ctg}\beta\Lambda}{\Lambda^2},\quad B_0^* = -2\left(\frac{\beta\operatorname{ctg}\beta\Lambda}{\Lambda}\right)^2$$
$$B_R^* = -\Lambda^{-1}[{}^1/_4({}^1/_2\varkappa_1^2 + \varkappa_2) + {}^1/_2\varkappa_1 L_1 + L_2]$$

Let us change back to the original coordinates in the system of Eqs.(4.7), putting $\mathbf{v}_\tau^a(\alpha_1, \alpha_2) = h\boldsymbol{\psi}(\alpha_1, \alpha_2)$. Omitting the O-term, we obtain

$$\frac{h^2}{2\Lambda}\Delta\mathbf{v}_\tau^a + (B + i\varepsilon B_0)\operatorname{grad}\operatorname{div}\mathbf{v}_\tau^a + \left[B_R + \left(\frac{\omega h}{c_2} - \Lambda\right)\right]\mathbf{v}_\tau^a = 0 \tag{4.8}$$
$$B = h^2 B^*,\quad B_0 = h^2 B_0^*,\quad B_R\mathbf{v}_\tau^a = -\frac{h^2}{\Lambda}\left\{\left[\frac{3}{8}\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right) + \frac{1}{4R_1R_2}\right]\mathbf{v}_\tau^a +\right.$$
$$\left.\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{v_1^a}{R_1}\mathbf{i}_1 + \frac{v_2^a}{R_2}\mathbf{i}_2\right) + \frac{v_1^a}{R_1^2}\mathbf{i}_1 + \frac{v_2^a}{R_2^2}\mathbf{i}_2\right\}$$

All the remarks made regarding Eq.(3.9) are equally applicable to Eq.(4.8). The two-dimensional system of equations for the case when the displacements u_k^0 define a symmetric SSS relative to the shell midsurface are obtained from (4.8) by replacing $\operatorname{ctg}\beta\Lambda$ with $-\operatorname{tg}\beta\Lambda$ in the expressions for B and B_0 and $n - {}^1/_2$ with n in the expression for Λ. In the case of forced oscillations, the right-hand side of (4.8) should contain the vector

$$\mathbf{P}_\tau = \pm\frac{(-1)^n h}{\Lambda}\left[\mathbf{F}_\tau + \frac{h\boldsymbol{\Phi}_\tau}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{4h\beta\operatorname{grad}F_3}{\Lambda}\left\{\begin{matrix}\operatorname{ctg}\beta\Lambda\\ -\operatorname{tg}\beta\Lambda\end{matrix}\right\}\right], \tag{4.9}$$
$$\Phi_j = {}^1/_2(Q_j^+ \mp Q_j^-)/E_*,\quad j = 1, 2,\quad \boldsymbol{\Phi}_\tau = \Phi_1\mathbf{i}_1 + \Phi_2\mathbf{i}_2$$

All the quantities in (4.9) and the rules for choosing the signs and the expressions in braces are the same as in (3.10). The expression for the external force vector (4.9) is identical with the corresponding expression for a dry shell.

5. A circular cylindrical shell. Let us discuss some qualitative features of SV HF SSS for the case of a circular cylindrical shell, when $R_1=\infty, R_2=R,$ and Eqs.(3.9) and (4.8) have constant coefficients. Let us consider type 1 SSS in more detail. We will seek particular solutions of (3.9) in the form

$$v_3^a=\exp\left[iR^{-1}(l_1\alpha_1+l_2\alpha_2)\right] \tag{5.1}$$

where l_1 and l_2 characterize the variability of the SSS along the midsurface coordinate lines.

Substituting (5.1) into Eq.(3.9), we obtain

$$\eta^2T^*(l_1^2+l_2^2)=\frac{2}{\Lambda}\eta^2\left(\frac{1}{16\beta^2}-1\right)+\frac{1}{2}i\varepsilon+\frac{1}{4}\varepsilon\eta\left(\frac{1}{\Lambda c}-i\right)+\left(\frac{\omega h}{c_2}-\Lambda\right) \tag{5.2}$$

Let $l_i\sim\eta^{-q_i}$, where $q_i\ (i=1,2)$, respectively, are the exponents of variability in the direction of the longitudinal axis and the circumference of the cylinder.

We will first consider the case when l_1 has to be determined given the oscillation frequency ω and the parameter l_2 characterizing the circumferential distribution of the solution. This formulation corresponds, for instance, to the problem of the forced oscillations of an immersed cylindrical shell under the action of an α_1-concentrated annular load with l_2 waves along the circumference /6/. Taking relationships (1.6) and (3.1) into account and using (5.2), we obtain

$$q_1=\max\left(q_2,\ 1-{}^1/_2b,\ 1-{}^1/_2t\right) \tag{5.3}$$

A number of important conclusions follow from (5.3). First, we can ignore the third term on the right-hand side of (5.2) with an error not exceeding $O(\eta)$, i.e., the piston theory may be used in this case to allow for the effect of the liquid. Second, if at least one of the three inequalities $q_2>0, b<2, t<2$ is satisfied, then with an error $O(\eta^{2q_1})$ we can ignore the first term on the right-hand side of formula (5.2) which determines the dependence of the solution on the exterior geometry of the cylinder (the principal radius of curvature).

Let us now determine the position of the real part of the l_2-th eigenfrequency for the plane problem $(l_1=0)$. This information is of interest for problems in scattering theory /7/. From Eq.(5.2) we have

$$\left(\frac{\omega h}{c_2}\right)_{l_2}=\Lambda+\eta^2\left[T^*l_2^2+\frac{2}{\Lambda}\left(1-\frac{1}{16\beta^2}\right)\right]-\frac{\varepsilon\eta}{4\Lambda c} \tag{5.4}$$

To fix our ideas we will assume that $t=1$ in (1.8), i.e., $\varepsilon\sim\eta$. Then the residual term in Eq.(3.8) is of order $O\left[\eta^{2-2q_2}(\eta+\eta^{2-2q_2})\right]$. The asymptotic formula (5.4) and the last estimate are identical with the results of /7/. For $q_2>0$, we can ignore the terms $O(\eta^2)$ on the right-hand side of (5.4), because the corresponding correction is asymptotically small compared with the distance between neighbouring resonances, which is of the order of $O(\eta^{2-q_2})$.

A similar analysis can be conducted for type 2 SV HF SSS. Without going into detail, we will merely note that unlike the previous case, when the imaginary term corresponding to the damping of the liquid was independent of the resonance index and had the constant order $O(\varepsilon)$, the order of this term for type 2 SSS is $O(\varepsilon\eta^{2-2q_2})$.

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STATIONARY QUASITRANSVERSE SIMPLE AND SHOCK WAVES IN A WEAKLY ANISOTROPIC NON-LINEAR ELASTIC MEDIUM*

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Two-dimensional stationary simple and shock waves in a weakly anisotropic non-linear elastic medium are considered under the same assumptions as in /1-6/, which studied one-dimensional non-stationary simple and shock waves in a prestrained non-linear elastic medium.

The standard analysis of stationary simple and shock waves /7-9/ in the magnetohydrodynamics of a gas with a frozen magnetic field essentially corresponds to a special case of an anisotropic elastic medium. Particular plane selfsimilar boundary-value problems of shock wave reflection from the boundary of an isotropic non-linear elastic half-space were solved numerically in /9, 10/.

1. Equations describing the behaviour of two-dimensional stationary simple waves. A weakly anisotropic non-linear elastic medium is defined by the elastic potential /1/

$$\Phi = \rho_0 U(\varepsilon_{ij}, p_l^{(k)} \ldots, \ldots, S), \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial \eta_j} + \frac{\partial w_j}{\partial \eta_i} + \frac{\partial w_k}{\partial \eta_i}\frac{\partial w_k}{\partial \eta_j}\right)$$

Here U is the internal energy of the medium, S is the entropy per unit mass, ε_{ij} are the components of Green's strain tensor, ρ_0 is the density in the unstressed state, $p_{l\ldots}^{(k)}$ are tensors specifying the deviation of the medium from an isotropic medium, w_i is the displacement vector and η_i are the Lagrangian coordinates (Cartesian right coordinates in the unstressed state); here and henceforth, $i, j, k = 1, 2, 3$.

The system of three equations of motion in Lagrangian Cartesian variables has the form /2/

$$\rho_0 \frac{\partial^2 w_i}{\partial t^2} = \frac{\partial}{\partial \eta_j} \frac{\partial \Phi}{\partial (\partial w_i/\partial \eta_j)} \tag{1.1}$$

and is of hyperbolic type.

We introduce a moving system of coordinates ξ_1, ξ_2, ξ_3 in which the motion of the system is steady.

$$\xi_1 = \eta_1 - |\mathbf{W}| t \sin\alpha, \ \xi_2 = \eta_2 - |\mathbf{W}| t \cos\alpha, \ \xi_3 = \eta_3$$

where $\mathbf{W}$ is a given vector of sufficiently large absolute value. The angle α defines the direction of the vector $\mathbf{W}$ relative to the axes η_1, η_2, η_3.

Let

$$\partial w_i/\partial \xi_1 = l_i, \ \partial w_i/\partial \xi_2 = m_i, \ \partial w_i/\partial \xi_3 = a_i$$

We assume that l_i, m_i, a_i are functions of the two variables ξ_1 and ξ_2. Therefore, we see from the equalities

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